BOUNDARY ENTROPY OF A HYPERBOLIC GROUP

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Abstract. We show that the entropy of a hyperbolic group acting on its ideal boundary is closely related to the exponential rate of its growth.

0. Introduction. Hyperbolic groups in the Gromov's sense [**Gro**] play an important role in geometric group theory (see [**GrH**] and the references there). In particular, any non-elementary hyperbolic group has exponential growth and the compact boundary of positive finite Hausdorff dimension ([**GhH**], pp. 126 and 157). Also, the group itself acts on its boundary *via* Lipschitz quasi-conformal maps (ibidem, p. 127). The dynamics of this action is of great interest. For instance, it has been shown [**CP**] that this action is finitely presented, i.e. it is semiconjugate to a subshift of finite type in such a way that the fibres of the conjugating map are finite of bounded length and the equivalence relation determined by this map (two points are related whenever their images are equal) is another subshift of finite type. Also, one can consider the topological entropy of this action in the sense of [**GLW**]. In this article we prove the following.

Theorem. *The topological entropy (with respect to a finite symmetric generating set) of a hyperbolic group G acting on its ideal boundary lies between the exponential rate of growth of G relative to suitable bounds depending on the geometry of the group and the exponential rate of growth of G (with respect to the same generating set).*

The precise description of the bounds mentioned in the Theorem can be found in Section 4 which contains also the proof of the Theorem and some final remarks. In the first section, we recall the notion of the topological entropy of a group action. In Section 2, we define the exponential rate of growth relative to given constants. In Section 3, we provide a short review on hyperbolic groups and spaces.

1. Entropy. Let G be a finitely generated group of homeomorphisms of a compact metric space (X, d) and S be a finite symmetric ($e \in S$, $S^{-1} = S$) set generating G. Equip G with the word metric d_S induced by S and let $B(n)$, $n \in \mathbb{N}$, denote the ball in G of radius n and centre e. Two points x and y of X are said to be (n, ϵ)-separated ($\epsilon > 0$, $n \in \mathbb{N}$) whenever

$$d(gx, gy) \geq \epsilon$$

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for some $g \in B(n)$. Since X is compact, the maximal number $N(n, \epsilon)$ of pairwise (n, ϵ)-separated points of X is finite. Also, there exist finite (n, ϵ)-spanning subsets of X: A subset A of X is (n, ϵ)-spanning whenever for any $y \in X$ there exists $x \in A$ such that

$$d(gx, gy) < \epsilon$$

for all $g \in B(n)$. Let $N'(n, \epsilon)$ denote the minimal cardinality of an (n, ϵ)-spanning subset of X.

Similarly to the case of classical dynamical systems ([**Wa**], p. 169), the families $N(n, \epsilon)$ and $N'(n, \epsilon)$ of functions have the same type of growth [**Eg**], more precisely, they have the same rate of exponential growth and the topological entropy $h(G, S)$ of G (w.r.t. S) can be defined by the formula

$$h(G, S) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N'(n, \epsilon).$$

If all the maps of G are Lipschitz and X has finite Hausdorff dimension, then $h(G, S)$ is finite for any S (compare [**GLW**], Prop.2.7). Also, if $h(G, S) = 0$ for some S, then $h(G, S') = 0$ for any other generating set S'. Therefore, one can distinguish between groups of positive and vanishing entropy without referring to generating sets.

2. Growth. Let us keep the notation of the previous section and recall that the exponential *rate of growth* of G (with respect to S) is defined as

$$\mathrm{gr}(G, S) = \lim_{n \to \infty} \frac{1}{n} \log N(n) = \lim_{n \to \infty} \frac{1}{n} N_0(n),$$

where $N(n) = \# B(n)$ and $N_0(n) = \# S(n)$ is the cardinality of the sphere $S(n)$ of radius n and centre e. Also, if $\epsilon > 0$ and $N_0(n; \epsilon)$ is the maximal cardinality of an ϵ-separated subset A of $S(n)$, then

$$\mathrm{gr}(G, S) = \lim_{n \to \infty} \frac{1}{n} \log N_0(n; \epsilon).$$

In fact, $N_0(n; \epsilon) \leq N_0(n)$ and, if A is such a subset of $S(n)$, then $\cup_{x \in A} B(x, \epsilon) \supset S(n)$ and, therefore, $N_0(n; \epsilon) N(\epsilon) \geq N_0(n)$ for any $n \in \mathbb{N}$.

Moreover, if $m \in \mathbb{N}$ and A_n is a maximal ϵ-separated subset of $S(mn)$, $n = 1, 2, \ldots$, then for any $x \in A_n$ we can find a sequence $(x_0, x_1, \ldots x_n)$ of elements of the group G for which $x_k \in A_k$, $x_n = x$, $x_0 = e$ and $d(x_k, x_{k+1}) \leq m + \epsilon$ for all k. To construct such a sequence one can begin with $x_n = x$, join x to e by a geodesic segment γ_x, find the point x'_{n-1} of intersection of γ_x with the sphere $S((n-1)m)$ and a point $x_{n-1} \in A_{n-1} \cap B(x'_{n-1}, \epsilon)$, and continue by the induction. If y is another point of A_n, $(y_0, y_1, \ldots, y_n)$ is a corresponding sequence and k is the maximal natural number such that $x_k = y_k$, then $d(x_{k+1}, y_{k+1}) \geq \epsilon$. This motivates the following definition.

Let us fix $m \in \mathbb{N}$, $\epsilon > 0$ and $\lambda \in (0, 1)$, and denote by $N_0(n; m, \epsilon, \lambda)$ the maximal cardinality of a subset A of $S(mn)$ satisfying the following condition:

$(*)$ If x and y lie in A, then there exist sequences $(x_0, x_1, \ldots x_n)$ and $(y_0, y_1, \ldots, y_n)$ of elements of G such that $x_k,\ y_k \in S(km)$, $x_0 = y_0 = e$, $x_n = x$, $y_n = y$,

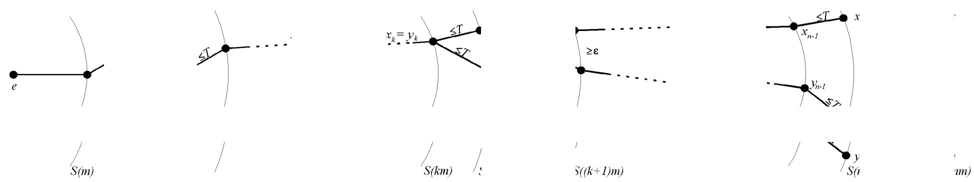

FIGURE 1.

$d(x_j, x_{j+1}) \leq T = m + \lambda\epsilon$ for all j and $d(x_{k+1}, y_{k+1}) \geq \epsilon$ when k is the maximal index for which $x_k = y_k$ (Figure 1).

The number

$$\mathrm{gr}^{\mathrm{rel}}(G, S; m, \epsilon, \lambda) = \limsup_{n\to\infty} \frac{1}{mn} \log N_0(n; m, \epsilon, \lambda)$$

will be called the exponential *rate of growth* of G *relative to* m, ϵ and λ. Finally, if $\mu : (0,1) \times \mathbb{R}_+ \to \mathbb{N}$ and $\tau : (0,1) \to \mathbb{R}_+$ are arbitrary functions, then we define the *rate of growth* of G *relative to* μ and τ by

$$\mathrm{gr}^{\mathrm{rel}}(G, S; \mu, \tau) = \sup\{\mathrm{gr}(G, S; m, \epsilon, \lambda); m > \mu(\lambda, \epsilon), \epsilon > \tau(\lambda), \lambda \in (0,1)\}.$$

Since $N_0(n; m, \epsilon, \lambda) \leq N_0(mn, \epsilon)$ for all m, n, ϵ and λ, we have

$$\mathrm{gr}^{\mathrm{rel}}(G, S; \mu, \tau) \leq \mathrm{gr}(G, S)$$

for all μ and τ as above. For the free group F_k generated by the set S_k of k free generators we have always

$$\mathrm{gr}^{\mathrm{rel}}(F_k, S_k; \mu, \tau) = \mathrm{gr}(F_k, S_k).$$

This is because F_k has no "dead ends" (see [**GrH**] for the definition and some information about some related problems) and in this case one can arrange ϵ-separated subsets A_n of the spheres $S(mn)$ in such a way that $\mathrm{dist}(x, A_n) = m$ for any $x \in A_{n+1}$. In general, one can expect that a relative rate of growth is strictly less than the "true" rate of growth.

3. Hyperbolic spaces and groups. Let (X, d) be a metric space. A curve $\gamma : [a, b] \to X$ is a *geodesic segment* when

$$d(\gamma(t), \gamma(s)) = |t - s|$$

for all $t, s \in [a, b]$. The space X is *geodesic* when any two points of X can be joined by a geodesic segment. For any finitely generated group G and any finite symmetric set S generating G, the Cayley graph $C(G, S)$ is geodesic.

Given three points x_0, y and z of a metric space X, the (based at x_0) *Gromov product* of y and z is given by

$$(y|z)_{x_0} = \frac{1}{2}\left(d(x_0, y) + d(x_0, z) - d(y, z)\right).$$

The space X is said to be *hyperbolic* (more precisely, δ-hyperbolic with $\delta \geq 0$) whenever the inequality

$$(x|z)_{x_0} \geq \min\{(x|y)_{x_0}, (y|z)_{x_0}\} - \delta$$

holds for arbitrary points x_0, x, y and z of X. Clearly, the Cayley graph of any free group F_k ($k = 1, 2, \dots$) generated by the set S_k of k free generators is a tree, so becomes 0-hyperbolic. A finitely generated group is said to be *hyperbolic* whenever its Cayley graph with respect to some (equiv., any) generating set is hyperbolic. Free groups and fundamental groups of compact Riemannian manifolds of negative sectional curvature are hyperbolic.

Assume that X is geodesic, take three points x_1, x_2 and x_3 of X and connecting them geodesic segments γ_1, γ_2 and γ_3. The union

$$\Delta = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

is a *geodesic triangle* with vertices x_i. The triangle Δ is *η-thin* ($\eta \geq 0$) when the canonical isometry f_Δ mapping Δ onto a tripod (i.e. the union of three segments with common origin) T_Δ satisfies the condition

$$d(x, y) \leq d(f_\Delta(x), f_\Delta(y)) + \eta$$

for all x and y of Δ.

In the proof of the Theorem we shall use the following.

Lemma 1. ([**GhH**], p. 41) *Let X be a geodesic metric space. If X is δ-hyperbolic, then all the geodesic triangles of X are 4δ-thin. Conversely, if all the geodesic triangles of X are η-thin, then X is 2η-hyperbolic.* $\square$

To construct the boundary ∂X of a hyperbolic space X let us fix a base point x_0 and say that a sequence (x_n) *diverges to infinity* whenever

$$\lim_{m,n\to\infty} (x_m|x_n) = \infty,$$

where $(\cdot|\cdot)$ denotes the Gromov product based at x_0. Two such sequences (x_n) and (y_m) are *equivalent* whenever

$$\lim_{m,n\to\infty} (x_m|y_n) = \infty.$$

The *boundary* ∂X of X consists of all the equivalence classes of sequences diverging to infinity. Note that ∂X can be described also in terms of equivalence classes of *geodesic rays* (i.e., maps $\gamma : [0, \infty) \to X$ such that $\gamma|[0, b]$ is a geodesic segment for any $b > 0$) or in terms of equivalence classes of *quasirays* (i.e. quasi-isometric maps of $[0, \infty)$ into X): Two such rays (or, quasirays) γ and σ are equivalent whenever their Hausdorff distance $d_H(\gamma, \sigma)$ is finite. The boundary point corresponding to the equivalence class of such γ is that determined by the sequence $x_n = \gamma(n)$, $n \in \mathbb{N}$. The equivalence of these constructions follows from the following fact which will be used later.

Lemma 2. ([**GhH**], p. 87) *Let X be a δ-hyperbolic geodesic metric space. For any $c \geq 1$ there exists $D \geq 0$ such that any c-quasigeodesic segment γ (i.e. any map $\gamma : [a,b] \to X$ (resp., $\gamma : [a,b] \cap \mathbb{Z} \to X$) such that the inequality $-c - c|s-t| \leq d(\gamma(s),\gamma(t)) \leq c|s-t| + c$ holds for all s and t of $[a,b]$ (resp., of $[a,b] \cap \mathbb{Z}$)) and any geodesic segment σ joining $\gamma(a)$ to $\gamma(b)$ satisfy the inequality*

$$d_H(\gamma,\sigma) \leq D. \quad \square$$

The set ∂X can be equipped with the metric structure as follows. First, for any ξ and ζ of ∂X put

$$(\xi|\zeta) = \sup \liminf_{m,n\to\infty} (x_m|y_n),$$

where (x_m) and (y_n) run over the set of all diverging to infinity sequences representing, respectively, ξ and ζ. Note that if X is δ-hyperbolic, then

$$(\xi|\zeta) - 2\delta \leq \liminf_{m,n\to\infty} (x_m|y_n) \leq (\xi|\zeta)$$

for all sequences (x_m) and (y_m) representing ξ and ζ. Next, choose $\eta > 0$ and put

$$\rho_\eta(\xi,\zeta) = \exp(-\eta \cdot (\xi|\zeta)).$$

Finally, let

$$d_\eta(\xi,\zeta) = \inf\{\sum_{i=0}^{k} \rho_\eta(\xi_i,\xi_{i+1}); \xi_i \in \partial X, \xi_0 = \xi \text{ and } \xi_{k+1} = \zeta, k \in \mathbb{N}\}.$$

If $\eta > 0$ is small enough, then d_η is a distance function on ∂X and $(\partial X, d_\eta)$ becomes a compact metric space of finite Hausdorff dimension ([**GhH**], pp. 122 - 126). Moreover, the inequalities

$$(1-2\eta')\rho_\eta(\xi,\zeta) \leq d_\eta(\xi,\zeta) \leq \rho_\eta(\xi,\zeta), \quad \xi,\zeta \in \partial X,$$

hold with

$$\eta' = \exp(\eta\delta) - 1.$$

4. Proof of the Theorem. Let again G be a finitely generated group, S - a finite symmetric set generating G and consider $C(G,S)$, the Cayley graph of G equipped with the distance function d satisfying

$$d(g_1,g_2) = |g_1^{-1}g_2|, \quad g_1,g_2 \in G,$$

where $|\cdot|$ is the length function on G determined by S, and making the edges of $C(G,S)$ isometric to Euclidean segments of length 1. Assume that $C(G,S)$ is δ-hyperbolic and let $\partial G = \partial C(G,S)$ be its boundary equipped, as in Section 3, with the distance function d_η, $\eta > 0$ being small enough.

The group G acts on $C(G,S)$ *via* isometries which, when restricted to $G \subset C(G,S)$, reduce to left translations L_g, $G \ni h \mapsto gh$. Therefore, each L_g extends to a Lipschitz homeomorphism, denoted by L_g again, of the boundary ∂G. Given

$\theta > 0$ denote by $N(n, \theta; \partial G)$ the maximal number of points of ∂G pairwise (n, θ)-separated by this action of G. Also, let $N'(n, \theta; \partial G)$ be the minimal cardinality of an (n, θ)-spanning subset of ∂G and $h(G, S; \partial G)$ - the corresponding entropy.

Since G is hyperbolic, there exists a constant $c_0 > 1$ such that for any $g \in G$ there exists $g' \in G$ such that $d(g, g') \le c_0$ and $|g'| = |g| + 1$ (see [**GrH**], p. 60). Let D be a corresponding constant such that any c_0-quasigeodesic segment lies at most D-apart (in the Hausdorff distance) from a true geodesic (compare Lemma 2). Let

$$\tau(\lambda) = \frac{4(D+\delta)}{1-\lambda} \quad \text{and} \quad \mu(\lambda, \epsilon) = \frac{\lambda\epsilon}{c_0 - 1}.$$

We shall show that

$$(**) \qquad \mathrm{gr}^{\mathrm{rel}}(G, S; \mu, \tau) \le h(G, S; \partial G) \le \mathrm{gr}(G, S).$$

We begin by the proof of the second inequality in $(**)$.

Choose $\theta > 0$ and a natural number k for which the inequality

$$\exp(-\eta \cdot k) < \theta$$

holds. For any $n \in \mathbb{N}$ and any $g \in S(n+k)$ choose, if only possible, a point $\xi_g \subset \partial G$ which can be connected to e by a geodesic ray, say γ_g, which passes through g. We claim that the set

$$A_n = \{\xi_g ; g \in S(n+k)\}$$

is (n, ϵ)-spanning in ∂G. Indeed, if $\zeta \in \partial G$, $\sigma : [0, \infty) \to X$ is a geodesic ray connecting e to ζ, $h \in S(n)$ and $g = \sigma(n+k)$, then $g \in S(n+k)$, the corresponding ray γ_g exists and satisfies the conditions

$$d(x_j, y_i) \le i + j - 2(n+k)$$

and

$$2(h^{-1}y_i | h^{-1}x_j) \ge (i-n) + (j-n) - (i+j) + 2(n+k) = 2k,$$

where $x_j = \sigma(j)$ and $y_i = \gamma(i)$ for all i and j sufficiently large (Figure 2).

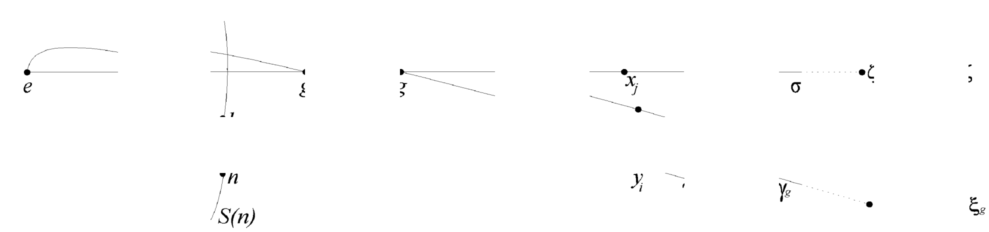

FIGURE 2.

Therefore,

$$d_\eta(L_{h^{-1}}\zeta, L_{h^{-1}}\xi_g) \le \rho_\eta(L_{h^{-1}}\zeta, L_{h^{-1}}\xi_g) < e^{-k\eta} < \theta.$$

This shows the inequalities

$$N'(n, \theta; \partial G) \le \# A_n \le \# S(n+k) \le N(n+k)$$

which imply immediately the required inequality in $(**)$.

The proof of the first inequality in $(**)$ is a bit more complicated.

Fix $\lambda \in (0,1)$, $\epsilon > \tau(\lambda)$ and $m > \mu(\lambda, \epsilon)$. For any $n \in \mathbb{N}$ choose a maximal subset A_n of $S(mn)$ satisfying condition $(*)$ of Section 2. For any $x \in A_n$ set $x_n = x$, choose a point $x_{n-1} \in A_{n-1}$ such that $d(x, x_{n-1}) \le m + \lambda\epsilon$, then a point $x_{n-2} \in A_{n-2}$ for which $d(x_{n-1}, x_{n-2}) \le m + \lambda\epsilon$ and so on. Finally, put $x_0 = e$. The map

$$\{0, m, \ldots, mn\} \ni j \mapsto x_{j/m}$$

is c-quasi-isometric with $c = (m + \lambda\epsilon)/m < c_0$.

Each map considered above can be extended to a c_0-quasi-isometric map

$$\gamma_x : \mathbb{N} \ni j \mapsto x'_j \in G$$

such that $x'_{im} = x_i$ for $i = 0, 1, \ldots, n$ and the sequence (x_j) converges to a point ξ_x of ∂G. We are going to show that the set

$$\{\xi_x; x \in A_n\}$$

is (n, θ)-separated under the action of G for some θ independent of n.

To this end, let us take arbitrary points x and y of A_n, $x \ne y$, choose sequences $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ as above, and let k be the maximal element of $\{0, 1, \ldots, n\}$ for which $x_k = y_k$. Denote this common value of x_k and y_k by g and consider the quasi-geodesic rays $\bar{\gamma}_x$ and $\bar{\gamma}_y$ obtained from γ_x and γ_y by restricting their domains to $\{mk, mk+1, \ldots\}$. Then, $\bar{\gamma}_x$ and $\bar{\gamma}_y$ originate at g and converge to ξ_x and ξ_y, respectively. By Lemma 2, there exist geodesic rays $\tilde{\gamma}_x$ and $\tilde{\gamma}_y$ originated at e and within the Hausdorff distance D from $L_{g^{-1}} \circ \bar{\gamma}_x$ and $L_{g^{-1}} \circ \bar{\gamma}_y$, respectively. Then, for any $j \in \mathbb{N}$ there exist positive real numbers s_j and t_j such that

$$d(\tilde{\gamma}_x(s_j), g^{-1}\bar{\gamma}_x(j)) \le D \quad \text{and} \quad d(\tilde{\gamma}_y(t_j), g^{-1}\bar{\gamma}_y(j)) \le D.$$

In particular,

$$d(\tilde{\gamma}_x(s_m), g^{-1}x_{k+1}) \le D \quad \text{and} \quad d(\tilde{\gamma}_y(t_m), g^{-1}y_{k+1}) \le D.$$

Clearly,

$$m - D \le s_m, t_m \le m + D + \lambda\epsilon, \ |s_m - t_m| \le 2D + \lambda\epsilon$$

and

$$d(\tilde{\gamma}_x(s_m), \tilde{\gamma}_y(t_m)) \ge \epsilon - 2D.$$

Assume that $(\tilde{\gamma}_x(s_j)|\tilde{\gamma}_y(t_j)) > m + D + \lambda\epsilon$ for some $j \in \mathbb{N}$. The isometry f_Δ corresponding to the geodesic triangle Δ with vertices $e, \tilde{\gamma}_x(s_j), \tilde{\gamma}_y(t_j)$ (compare

Lemma 1) maps the points $\tilde{\gamma}_x(s_m)$ and $\tilde{\gamma}_y(t_m)$ onto some points of the originated at e edge of the tripod $f_\Delta(\Delta)$ and, therefore, satisfies the condition

$$\begin{aligned} d(\tilde{\gamma}_x(s_m), \tilde{\gamma}_y(t_m)) &\le d(f_\Delta(\tilde{\gamma}_x(s_m)), f_\Delta(\tilde{\gamma}_y(t_m))) + 4\delta \\ &= |s_m - t_m| + 4\delta \le 2D + \lambda\epsilon + 4\delta < \epsilon - 2D. \end{aligned}$$

Comparing the inequalities above we obtain a contradiction which shows that

$$(\tilde{\gamma}_x(s_j)|\tilde{\gamma}_y(t_j)) \le m + D + \epsilon\lambda$$

for all $j \in \mathbb{N}$. This inequality proves that

$$\begin{aligned} d_\eta(g^{-1}\xi_x, g^{-1}\xi_y) &\ge (1 - 2\eta')\rho_\eta(g^{-1}\xi_x, g^{-1}\xi_y) \\ &\ge (1 - 2\eta')\exp(-\eta(m + D + \lambda\epsilon + 2\delta)), \end{aligned}$$

i.e. that the points ξ_x and ξ_y are (nm, θ)-separated with

$$\theta = (1 - 2\eta')\exp(-\eta(m + D + \lambda\epsilon + 2\delta)).$$

The above argument implies the inequality

$$N(nm, \theta; \partial G) \ge \# A_n \ge N_0(G, S; m, \epsilon, \lambda).$$

which holds for all n. Passing to suitable limits when $n \to \infty$ yields the required inequality in $(**)$. $\square$

In [**Fr**], Friedland defined the *minimal entropy* $h_{\min}(G)$ of a finitely generated group G of homeomorphisms of a compact metric space X:

$$h_{\min}(G) = \inf_S h(G, S),$$

where S ranges over all finite symmetric sets generating G. Similarly, the *minimal rate of growth* $\mathrm{gr}_{\min}(G)$ of any finitely generated group G can be defined as follows (compare [**GrH**]):

$$\mathrm{gr}_{\min}(G) = \inf_S \mathrm{gr}(G, S).$$

The reader can define the minimal relative rate of growth $\mathrm{gr}^{\mathrm{rel}}_{\min}(G)$ appropriately.

If G is hyperbolic, then id_G induces a Hölder homeomorphism of boundaries of G obtained from different generating sets ([**GhH**], page 128). Therefore, the boundary entropy of such G (w.r.t. a given finite symmetric generating set S) does not depend on the choice of a generating set used in the construction of ∂G and our Theorem implies immediately the following.

Corollary. *For any hyperbolic group G the equalities*

$$\mathrm{gr}^{\mathrm{rel}}_{\min}(G) \le h_{\min}(G, \partial G) \le \mathrm{gr}_{\min}(G)$$

hold. $\square$

This answers partially the following question asked by Friedland in [**Fr**]: Find a geometric interpretation of the minimal entropy of a Kleinian group acting on the ideal boundary of a hyperbolic space H^n.

For the free group F_k with k generators the above discussion and an argument of [**GLP**] (p. 70) imply the equality

$$h_{\min}(F_k, \partial F_k) = \mathrm{gr}_{\min}(F_k) = \mathrm{gr}^{\mathrm{rel}}_{\min}(F_k) = \log(2k-1).$$

In fact, if S is any finite symmetric set generating F_k, then the elements of S represent members of a set S' generating $\mathbb{Z}^k$, the abelianization of F_k. S' contains a symmetric set R' such that $\#R' = 2k$ and the subgroup of $\mathbb{Z}^k$ generated by R' has finite index. The corresponding subset R of S consists also of $2k$ elements and generates the free group isomorphic to F_k. Therefore,

$$\begin{aligned} h(F_k, S; \partial F_k) \geq h(F_k, R; \partial F_k) &= h(F_k, S_k; \partial F_k) \\ &\geq \mathrm{gr}^{\mathrm{rel}}(F_k, S^k) = \mathrm{gr}(F_k, S_k) = \log(2k-1). \end{aligned}$$

The opposite inequality is obvious.

The other natural case, that of the fundamental group Γ_g of a closed oriented surface of genus $g > 1$ is more complicated: The minimal rate of growth of Γ_g is still unknown even if some estimates exist: $\mathrm{gr}_{\min}(\Gamma_g) \geq 4g-3$, $\mathrm{gr}_{\min}(\Gamma_g) \leq \mathrm{gr}(\Gamma_g, S_g) \approx 4g - 1 - \epsilon_g$, where S_g is the canonical set of generators of Γ_g and ϵ_g is a small constant found numerically. In particular, $5 \leq \mathrm{gr}_{\min}(\Gamma_2) \leq 6.9798$. The calculation or estimation of the value of the minimal relative rate of growth of Γ_g is yet more difficult.

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